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'Analogue of Hartogs' Lemma for Pluriharmonic Functions

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Abstract: In this paper, an analogue of the Hartogs' lemma for pluriharmonic functions is proved. The conditions of the theorem are, of course, necessary for the pluriharmonicity of functions of many variables. Conditions 1, characterizing the harmonicity of a function in direction z_n is standard in theorems of continuation along lines. It defines the principle of continuation of a function in a direction.

Keywords: Holomorphic function, real-analytic function, harmonic functions, subharmonic functions, plurisubharmonic functions, pluriharmonic functions.

INTRODUCTION

Pluriharmonic functions play an important role in the theory of functions of many complex variables. Being a real part of analytic functions, they are often used in studies of geometric properties of the class of analytic functions, in the theory of integral formulas, in complex potential theory, and others ([Ronkin, L. I, 1971]-[Madrakhimov, R. M. *et al*., 2001]).

In this article we consider the issues of continuation of pluriharmonic functions along the direction of the axis OZ_n. An analogue of the Hartogs lemma for pluriharmonic functions is proved.

Theorem: Let the function $U({}^{/}z, z_n)$ defined in the polycircle ${}^{/}V \times \{|z_n| < R\} \subset C^{n-1}_{{}^{/}z} \times C_{z_n}, R > 0$ and satisfies the following conditions: For each fixed ${}^{/}z \in {}^{/}V$ function $U({}^{/}z, z_n)$ complex variable z_n harmonious in the circle $|z_n| < R$

1) $U({}^{/}z, 0)$ harmonious in ${}^{/}V$ by ${}^{/}z$
2) function $U({}^{/}z, z_n)$ m- subgamonic (plurisubharmonic) in some polycircle ${}^{/}V \times \{|z_n| < r\},\ R > r > 0$
3) Then the function $U({}^{/}z, z_n)$ pluriharmonic in a polycircle ${}^{/}V \times \{|z_n| < R\}$

To prove the theorem we need the following lemma, which is a harmonic analogue of this theorem.

Lemma. Let the function $U({}^{/}z, z_n)$ defined in the polycircle ${}^{/}V \times V_n$ и satisfies the following conditions:

1) For each fixed ${}^{/}z \in {}^{/}V$ function $U({}^{/}z, z_n)$ variable z_n harmonious in the circle V_n
2) Function $U({}^{/}z, 0)$ harmonious in ${}^{/}V$ по ${}^{/}z$
3) Function $U({}^{/}z, z_n)$ subharmonic in a polycircle ${}^{/}V \times V_n$
4) Then the function $U({}^{/}z, z_n)$ harmonious in a polycircle ${}^{/}V \times V_n$.

The conditions of the theorem are, of course, necessary for the pluriharmonicity of functions of many variables. Conditions 1, characterizing the harmonicity of a function in direction z_n is standard in theorems of continuation along lines. It defines the principle of continuation of a function in a direction. Note that conditions 2 and 3 are also essential: function $U(z_1, z_2)\ = x_1 x_2$ in a circle $V = \{|z_1| < 1, |z_2| < 1\}$ satisfies conditions 1,2, however it does not satisfy condition 3, and the function

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$U(z_1,z_2) = |Z_1|^2$ satisfies conditions 1.3, but it is also not pluriharmonic in $V = \{|z_1| < 1, |z_2| < 1\}$ (condition 2 is not met).

Proof. Let's take an arbitrary number $r < R$ and for fixed ${}^{/}z \in {}^{/}V$ и $|z_n| < r$ according to the Poisson formula we express $U({}^{/}z, z_n)$ via the integral over a circle,

$$U({}^{/}z, z_n) = \frac{1}{2\pi}\int_0^{2\pi} U({}^{/}z, \xi)\,\mathrm{Re}(\frac{\xi + z_n}{\xi - z_n})dt \text{ (I)}, \quad \text{where } \xi = re^{it}$$

First, for the sake of clarity of presentation, let us consider the case when $U \in C^2({}^{/}V \times V_n)$, i.e. U is a twice smooth function with respect to the set of variables.

By the condition of the lemma, the Laplace operator

$$\Delta U = \Delta_{{}^{/}z} U + \Delta_{z_n} U = \Delta_z U \geq 0$$

Because U harmonious in z_n, and therefore $\Delta_{z_n} U = 0$ at any fixed ${}^{/}z \in {}^{/}V$. From here and from formula (I) we have

$$\Delta U = \Delta_{{}^{/}z} U = \frac{1}{2\pi}\int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi)\,\mathrm{Re}(\frac{\xi + z_n}{\xi - z_n})dt \geq 0$$

Let's consider the function,

$$\psi({}^{/}z, z_n) = \frac{1}{2\pi}\int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi)\frac{\xi + z_n}{\xi - z_n}dt$$

the real part of which coincides with ΔU. It is clear that it is holomorphic in z_n in a circle ${}^{/}V \times \{|z_n| < r\}$ for any fixed ${}'Z \in {}^{/}V$. Now, from condition 3 of the lemma it follows that the Laplace operator $\Delta_{{}^{/}z} U({}^{/}z, 0) = 0$

According to the mean value theorem (which also follows from the expression (I) при $z_n = 0$) we have

$$\Delta_{{}^{/}z} U({}^{/}z, 0) = \frac{1}{2\pi}\int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi)dt$$

From here $\psi({}^{/}z, 0) = \frac{1}{2\pi}\int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi)dt = \Delta_{{}^{/}z} U({}^{/}z, 0) = 0$

From the property of openness of holomorphic functions, holomorphic in z_n function $\psi({}^{/}z, z_n)$ either $\equiv 0$, or maps a neighborhood of zero to some neighborhood of zero. The latter is impossible due to the fact that

$$\mathrm{Re}\,\psi = \Delta U \geq 0 \quad \text{в} \quad {}^{/}V \times \{|z_n| < r\}$$

Hence, $\psi \equiv 0$. Thus $\Delta U = 0$ in ${}^{/}V \times \{|z_n| < r\}$. Because, $r < R$ arbitrary, then from here we get the harmonic function $U({}^{/}z, z_n)$ in ${}^{/}V \times V_n$.

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Now let us consider the general case when is an arbitrary subharmonic function.

By the condition of the lemma, in this case the Laplacian ΔU the functional defined by the integral is positive in the generalized sense

$$(\Delta U,\varphi)=(U,\Delta\psi)=\int\Delta U\varphi dV$$

where φ - infinitely smooth, finite in a polycircle ${}^{/}V\times V_n$ function is positive, i.e. $(\Delta U,\varphi)\geq 0$ at $\varphi\geq 0$. In particular, for any finite non-negative functions $\varphi_1({}^{/}z)$, $\varphi_2(z_n)$ class C^{∞} respectively in the areas ${}^{/}V$ and V_n we have

$$\left(U,\Delta\left(\varphi_1\cdot\varphi_2\right)\right)=\int\limits_{{}^{/}V\times V_n}U\left({}^{/}z,z_n\right)\Delta\left(\varphi_1({}^{/}z),\varphi_2(z_n)\right)dV\geq 0$$

where dV - volume forms in C^n. Because, $\Delta(\varphi_1\cdot\varphi_2)=(\Delta_{{}^{/}z}\varphi_1)\varphi_2+\varphi_1(\Delta z_n\varphi_2)$

Then, according to Fubini's theorem and from harmonicity U по z_n we have

$$\left(U,\Delta\left(\varphi_1\cdot\varphi_2\right)\right)=\int\limits_{V_n}\left\{\int\limits_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_1 d^{/}V\right\}\varphi_2 dV_n+\int\limits_{V_n}\left\{\int\limits_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_2 dV_n\right\}\varphi_1 d^{/}V=$$

$$=\int\limits_{V_n}\left\{\int\limits_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_1 d^{/}V\right\}\varphi_2 dV_n$$

where $d^{/}V$ and dV_n - volume formulas in $C^{n-1}_{{}^{/}z}$ and C_{z_n} respectively.
From here

$$\left(U,\Delta\left(\varphi_1\cdot\varphi_2\right)\right)=\int\limits_{V_n}\left\{\int\limits_{{}^{/}V}\Delta\varphi_1 d^{/}V\right\}\varphi_2 dV_n\geq 0$$

For all finite non-negative functions $\varphi_2(z_n)$.

Therefore, the inner integral is positive:

$$\int\limits_{{}^{/}V}U({}^{/}z,z_n)\Delta\varphi_1({}^{/}z)d^{/}V$$

For each fixed $z_n\in V_n$

Hence, by Fubini's theorem and their formula (I) for each fixed $r<R$ and for each fixed $z_n\in V_n^{/}=\left\{\left|z_n\right|<r\right\}$

We have

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$$(U,\Delta\varphi_1)=\int_{{}^{/}V}\left\{\frac{1}{2\pi}\int_0^{2\pi}U({}^{/}z,\xi)\,\mathrm{Re}(\frac{\xi+z_n}{\xi-z_n})dt\right\}\Delta\varphi_1 d^{/}V=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{{}^{/}V}U({}^{/}z,\xi)\Delta\varphi_1 d^{/}V\right\}\mathrm{Re}(\frac{\xi+z_n}{\xi-z_n})dt\geq 0$$

Where $\xi=re^{it}$

It is clear that the function

$$\psi(z_n)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{{}^{/}V}U({}^{/}z,\xi)\Delta\varphi_1 d^{/}V\right\}\frac{\xi+z_n}{\xi-z_n}dt$$

holomorphic in z_n at $V_n^{/}$, and

$$\psi(0)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{{}^{/}V}U({}^{/}z,\xi)\Delta\varphi_1 d^{/}V\right\}dt=\int_{{}^{/}V}U({}^{/}z,0)\Delta\varphi_1({}^{/}z)d^{/}V=0$$

According to the mean value theorem and the condition of the lemma.

Because

$$(U,\Delta\varphi_1)=\mathrm{Re}\psi\geq 0$$

In $V_n^{/}$, then it follows from this that $\psi\equiv 0$

From here

$$\int_{{}^{/}V}U({}^{/}z,z_n)\Delta\varphi_1 d^{/}V=0$$

For each fixed $|z_n|<r$ and for any non-negative in ${}^{/}V$ function φ_1. Therefore, for every fixed $z_n\in V_n$ function $U({}^{/}z,z_n)$ harmonious in ${}^{/}V$ by ${}^{/}z$. Now it is easy to show that $\Delta U=0$ in a generalized sense. Indeed, for any finite in the polycircle ${}^{/}V\times V_n$ function ψ we have:

$$\left(U,\Delta\psi\right)=\int U\Delta\psi dV=\int U(\Delta_{{}^{/}z}\psi+\Delta_{z_n}\psi)dV=$$

$$=\int U\Delta_{{}^{/}z}\psi dV+\int U\Delta_{z_n}\psi dV=0$$

Therefore, the Laplace operator ΔU from the subharmonic function U equals zero in the general sense. Such a subharmonic function must be harmonic according to the Riesz representation. The lemma is proved.

Let us proceed to the proof of the theorem. It follows from the proven lemma and from the following fact: if the function U - simultaneously harmonic and plurisubharmonic in the polycircle ${}^{/}V\times V_n$, then it is pluriharmonic in the polycircle ${}^{/}V\times V_n$. (см[1])

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Besides this, there is the following fact: if the function U - simultaneously harmonic [9] and m-subharmonic in the polycircle ${}^{/}V \times V_n$, then it is pluriharmonic in the polycircle ${}^{/}V \times V_n$.

From the conditions of Theorem I, according to Lemma I, it follows that $U({}^{/}z, z_n)$ harmonious in a polycircle ${}^{/}V \times \{|z_n| < r\}$.

CONCLUSION

Then from plurisubharmonicity U in this polycircle it follows that the function U is pluriharmonic in ${}^{/}V \times \{|z_n| < r\}$. From the analogue of the Hartogs lemma for pluriharmonic functions we obtain that U pluriharmonic in a large polycircle ${}^{/}V \times \{|z_n| < R\}$. The theorem is proven.

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