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# Poisson Chi-Square Mixture Distribution with Graphical Simulatio at some Parameter Values ν and λ

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**Abstract:** The wide range of applications of mixture of two probability distributions have been known in the literature. Because of its discovered importance in modelling problems, research on the development of more probability distribution remains an interesting area. This study defines a new probability distribution that is defined as Poisson Chi-Square Mixture. A proof that the newly introduced distribution is indeed a probability distribution is presented by satisfying the basic properties. Properties such as mean, variance, skewness, and kurtosis are derived using cumulants. Further, simulation using Maple software was done to visualize to shape of the Poisson Chi-Square Mixture at the different values of parameter values ν and λ. This study however is only limited to introduce a new probability distribution and data fitting is recommended for future researchers. Major findings provide additional list of probability distribution.

**Keywords:** Poisson Chi-Square Mixture, Mixture Distribution, Graphical Simulation.

## INTRODUCTION

Mixture distributions in general have known various applications particularly in modelling population in medicine, industry and economics (Nasiri and Azarian, 2018). Wang *et al*. (2014) used this models for modelling time-to-event data to evaluate treatment effects in randomized clinical trials. Zaman (2006a) introduced the chi-square mixture of chi-square distribution. Hory and Martin (2003) proposed a mixture of chi-square distributions to illustrate the time distribution of light in the imaging. In similar year, Hory and Martin (2003) used chi-square mixture distributions to describe an unstructured distribution model for distributing light's distribution time.

In this study, a new chi-square mixture distribution is introduced. Using the definition of Zaman *et al*. (2006a), the Poisson Chi-Square Mixture Distribution is defined as

$$fx(x;v,\lambda) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{e^{-(\lambda+\chi^2)}(\lambda+\chi^2)^x}{x!} d\chi^2 \qquad (1)$$

$$\text{for } 0 < x < \infty.$$

To show that (1) is a probability distribution, this paper provides computational proof that the integral of (1) for all x results to 1. Further, the newly defined distribution is simulated using maple software at different values of ν degrees of freedom and parameter λ.

### Computational Proof and Derivation

To proceed with the proof, although not presented here, it is necessary to recall definitions of gamma function, poison distribution, and the chi-square distribution.

Take note that (1) is non-negative since chi-square and Poisson are non-negative probability distributions. Further,





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$$\begin{aligned}\int_0^\infty f_X(x,\lambda,v)dx &= \int_0^\infty \left[\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{e^{-(\lambda+\chi^2)}(\lambda+\chi^2)^x}{x!}d\chi^2\right]dx\\
&= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\sum_{x=0}^\infty \frac{e^{-(\lambda+\chi^2)}(\lambda+\chi^2)^x}{x!}d\chi^2\\
&= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}e^{-(\lambda+\chi^2)}\sum_{x=0}^\infty \frac{(\lambda+\chi^2)^x}{x!}d\chi^2\\
&= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{e^{-(\lambda+\chi^2)}}{e^{-(\lambda+\chi^2)}}d\chi^2\end{aligned}$$

**Let $u = \frac{\chi^2}{2}$ so that $du = \frac{1}{2}d\chi^2$. Thus,**

$$\begin{aligned}\int_0^\infty f_X(x,\lambda,v)dx &= \lim_{c\to\infty}\left[\int_0^\infty \frac{e^u(2u)^{\frac{v}{2}-1}2du}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\right]\\
&= \frac{2^{\frac{v}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\int_0^\infty e^u u^{\frac{v}{2}-1}du\\
&= \frac{2^{\frac{v}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\Gamma(\frac{v}{2})\\
&= 1.\end{aligned}$$

**That is, $\int_0^\infty f_X(x,\lambda,v)dx = 1$.**

The characteristic function of the Poisson Chi-Square Mixture distribution is derived using the definition of (Zaman *et al*., 2006b) is given by;

$$\begin{aligned}\Phi_x(t) &= \mathbb{E}[e^{itx}]\\
&= \int_0^\infty e^{itx}\left[\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{e^{-(\lambda+\chi^2)}(\lambda+\chi^2)^x}{x!}d\chi^2\right]dx\\
&= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\sum_{x=0}^\infty e^{itx}\frac{e^{-(\lambda+\chi^2)}(\lambda+\chi^2)^x}{x!}d\chi^2\\
&= \lim_{c\to\infty}\left[\int_0^c \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}e^{-(\lambda+\chi^2)}\sum_{x=0}^\infty \frac{(e^{it}(\lambda+\chi^2))^x}{x!}d\chi^2\right]\\
&= \lim_{c\to\infty}\int_0^c \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}e^{-(\lambda+\chi^2)}(e^{(e^{(it)})(\lambda+\chi^2)})}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}d\chi^2\\
&= \lim_{c\to\infty}\int_0^c \frac{e^{(\lambda e^{it}-\lambda)+(\chi^2e^{it}-\frac{3\chi^2}{2})}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}d\chi^2\\
&= \lim_{c\to\infty}\int_0^c \frac{e^{(\lambda e^{it}-\lambda)}e^{-\chi^2(\frac{3}{2}-e^{it})}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}d\chi^2\\
&= \frac{e^{(\lambda e^{it}-\lambda)}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\lim_{c\to\infty}\int_0^c e^{-\chi^2(\frac{3}{2}-e^{it})}(\chi^2)^{\frac{v}{2}-1}d\chi^2\end{aligned}$$



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Using the method of moments (Zaman *et al.*, 2006b), the cumulant generating function is expressed as;

Therefore, the Taylor series expansion is given by;

$$\ln(3-2e^{it}) = -2(it) - 6\frac{(it)^2}{2!} - 30\frac{(it)^3}{3!} - 210\frac{it^4}{4!} + \ldots$$

$$-\frac{v}{2}\ln(3-2e^{it}) = v(it) + 3v\frac{(it)^2}{2!} + 15\frac{(it)^3}{3!} + 105\frac{it^4}{4!} + \ldots$$

$$\kappa_x(t) = \lambda(1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{it^4}{4!} + \ldots) - \lambda + (v(it) + 3v\frac{(it)^2}{2!} + 15\frac{(it)^3}{3!} + 105\frac{it^4}{4!} + \ldots)$$
$$= (\lambda + v)(it) + (\lambda + 3v)\frac{(it)^2}{2!} + (\lambda + 15v)\frac{(it)^3}{3!} + (\lambda + 105v)\frac{it^4}{4!} + \ldots$$

Hence the mean is

$$\mu_1' = \kappa_1 = (\lambda + v),$$

Also, the variance is given by

$$\mu_2 = \kappa_2 = (\lambda + 3v),$$

The third central moment is

$$\kappa_3 = \mu_3 = (\lambda + 15v),$$

Finally, the fourth central moment

$$\mu_4 = \kappa_4 + 3{\kappa_2}^2$$

$$= (\lambda + 105v) + 3(\lambda + 3v)^2$$

$$= \lambda + 105v + 3(\lambda^2 + 6\lambda v + 9v^2)$$

$$= \lambda + 105v + 3\lambda^2 + 18\lambda v + 27v^2$$

For the skewness

$$\beta_1 = \frac{(\kappa_2)^2}{(\kappa_2)^3}$$
$$= \frac{\mu_3^2}{\mu_2^3}$$
$$= \frac{(\lambda + 15v)^2}{(\lambda + 3v)^2}$$

While kurtosis is given by

$$\beta_2 = \frac{3(\kappa_2)^2 + \mu_4}{(\kappa_2)^2}$$
$$= \frac{3(\lambda + 3v)^2 + (\lambda + 105v)}{(\lambda + 3v)^2}$$



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Note further that both $\beta_1$ and $\beta_2$ are always positive, this implies that the Poisson Chi-Square Mixture distribution is skewed to the right and leptokurtic.

**Graphical Simulation of Poisson Chi-Square Mixture distributions**

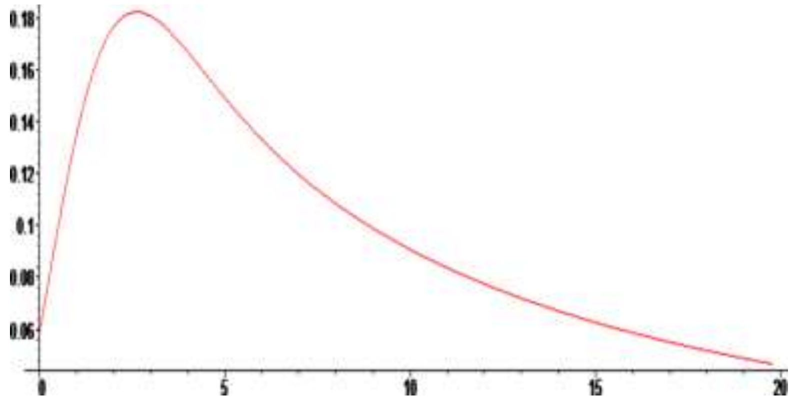


**Figure 1**. Shape of Possion Chi-Square Mixture with $\nu = 1$ and $\lambda = 1.5$

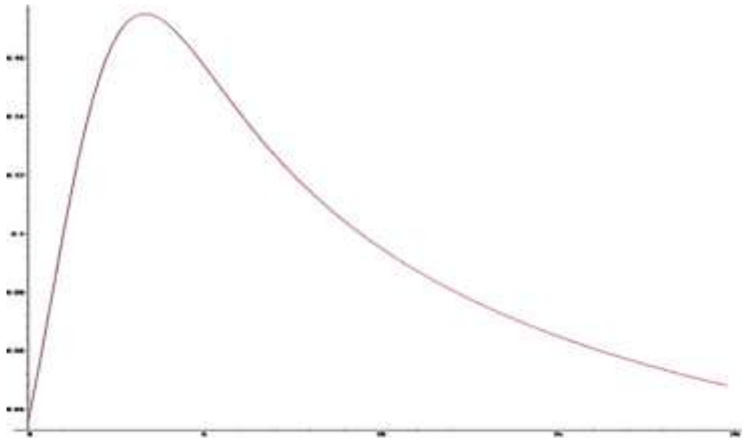

**Figure 2**. Shape of Possion Chi-Square Mixture with $\nu = 1$ and $\lambda = 2$

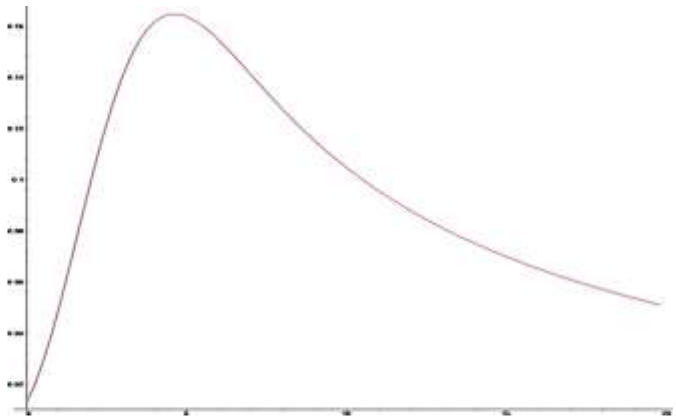

**Figure 3**. Shape of Possion Chi-Square Mixture with $\nu = 1$ and $\lambda = 3$

It is observed in the given figures that the distribution seemed to lift its right tale when $\lambda$ increases while fixing the parameter $\nu$. Also, all figures support the results showing that the newly defined distribution is skewed to the right and leptokurtic.

## CONCLUSION

This study presented a proof that the newly defined Poisson Chi-Square Mixture distribution is indeed a probability distribution. Furthermore, it is found out that in most of the time, the distribution is both skewed to the right and leptokurtic. Major findings theoretically add new entry in the existing



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list of known probability distributions whose applications can be further investigated. Moreover, the researchers highly recommended for the model fitting of the Poisson Chi-Square Mixture distribution using actual and empirical data.

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