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Lorentz Mapping, Complex Unimodular Matrix and Dirac Spinor

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Abstract: For a given Lorentz mapping, we deduce the corresponding 2 x 2 unimodular complex matrix, and the transformation of the Dirac spinor.
Keywords: Unimodular complex matrix, Dirac spinor, Lorentz matrix.

INTRODUCTION

The arbitrary complex quantities α, β, γ, δ verifying the condition $\alpha\delta - \beta\gamma = 1$, generate a Lorentz matrix $L = (L^{\mu}{}_{\nu})$ via the expressions [Rumer, J. 1936 - Cruz-Santiago, R. *et al.*, 2021]:

$$L^0{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \qquad L^1{}_0 = \frac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, \qquad L^2{}_0 = -\frac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc,$$

$$L^0{}_1 = \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \qquad L^1{}_1 = \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \qquad L^2{}_1 = -\frac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,$$

$$L^0{}_2 = -\frac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \qquad L^1{}_2 = -\frac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \qquad L^2{}_2 = \frac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc, \tag{1}$$

$$L^0{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), \qquad L^1{}_3 = \frac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, \qquad L^2{}_3 = -\frac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc,$$

$$L^3{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \qquad L^3{}_1 = \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \qquad L^3{}_2 = -\frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc,$$

$$L^3{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \qquad \alpha\delta - \beta\gamma = 1,$$

Where *cc* means the complex conjugate of all the previous terms.

The inverse problem is to obtain $\alpha, \beta, \gamma, \delta$ if we know L, and the answer is [Cruz-Santiago, R. *et al.*, 2021- López-Bonilla, J. *et al.*, 2021]:

$$\alpha = \frac{1}{D}\,Q^1{}_1 = \frac{1}{2D}[L^0{}_0 + L^0{}_3 + L^1{}_1 + L^2{}_2 + L^3{}_0 + L^3{}_3 - i\,(L^1{}_2 - L^2{}_1)],$$

$$\beta = \frac{1}{D}\,Q^1{}_2 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 - L^1{}_3 + L^3{}_1 + i\,(L^0{}_2 + L^2{}_0 - L^2{}_3 + L^3{}_2)], \tag{2}$$

$$\gamma = \frac{1}{D}\,Q^2{}_1 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 + L^1{}_3 - L^3{}_1 - i\,(L^0{}_2 + L^2{}_0 + L^2{}_3 - L^3{}_2)],$$

$$\delta = \frac{1}{D}\,Q^2{}_2 = \frac{1}{2D}[L^0{}_0 - L^0{}_3 + L^1{}_1 + L^2{}_2 - L^3{}_0 + L^3{}_3 + i\,(L^1{}_2 - L^2{}_1)],$$

$$\text{where } D^2 = Q^1{}_1\,Q^2{}_2 - Q^1{}_2\,Q^2{}_1 .$$

On the other hand, the Dirac spinor obeys the transformation law [Leite-Lopes, J. 1977; Ohlsson, O. 2011]:

$$\tilde{\psi} = S\,\psi . \tag{3}$$

For a non-singular matrix S such that:

$$L^{\mu}{}_{\nu}\,S\,\gamma^{\nu} = \gamma^{\mu}\,S\,, \tag{4}$$

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And we must determine a solution of (4) for a given Lorentz transformation. We have the expansion [Caicedo-Ortiz, H. E, J. *et al.*, 2021]:

$$S = b_0\, I + i d_0\, \gamma^5 + b_1\, \sigma^{23} + b_2\, \sigma^{31} + b_3\, \sigma^{12} + \sum_{j=1}^{3} d_j\, \sigma^{oj},$$

$$b_0^2 - d_0^2 + \sum_{j=1}^{3} (d_j^2 - b_j^2) = 1, \qquad b_0\, d_0 - \sum_{j=1}^{3} b_j\, d_j = 0, \tag{5}$$

in terms of Dirac matrices in the standard representation [Leite-Lopes, J. 1977].

From (4) are immediate the expressions [Ohlsson, O. 2011; Macfarlane, A. J. 1966]:

$$L^{\mu}{}_{0} = \frac{1}{4}\, tr\, (\gamma^0\, S^{-1}\, \gamma^{\mu}\, S), \qquad L^{\mu}{}_{k} = -\frac{1}{4}\, tr\, \left(\gamma^k\, S^{-1}\, \gamma^{\mu}\, S\right), \quad \mu = 0, \dots, 3, \quad k = 1, 2, 3, \tag{6}$$

That is, if we know *S* then with (6) we can determine the Lorentz matrix; (6) generates the relations:

$$L^0{}_0 = 2(b_0^2 - b_1^2 - b_2^2 - b_3^2) - 1, \qquad L^0{}_1 = 2[\, (b_2\, d_3 - b_3\, d_2) + i\, (b_0\, d_1 - b_1\, d_0)\,],$$
$$L^0{}_2 = 2[\, (b_3\, d_1 - b_1\, d_3) + i\, (b_0\, d_2 - b_2\, d_0)\,], \qquad L^0{}_3 = 2[\, (b_1\, d_2 - b_2\, d_1) + i\, (b_0\, d_3 - b_3\, d_0)\,],$$
$$L^1{}_0 = 2[-\, (b_2\, d_3 - b_3\, d_2) + i\, (b_0\, d_1 - b_1\, d_0)\,], \quad L^1{}_1 = 2[\, (b_0^2 - b_1^2) + (d_2^2 + d_3^2)\,] - 1,$$
$$L^1{}_2 = 2[-(b_1\, b_2 + d_1\, d_2) - i\, (b_0\, b_3 + d_0\, d_3)\,], \quad L^1{}_3 = 2[-\, (b_1\, b_3 + d_1\, d_3) + i\, (b_0\, b_2 + d_0\, d_2)\,],$$
$$(7)$$
$$L^2{}_0 = 2[-(b_3\, d_1 - b_1\, d_3) + i\, (b_0\, d_2 - b_2\, d_0)\,], \quad L^2{}_1 = 2[-\, (b_1\, b_2 + d_1\, d_2) + i\, (b_0\, b_3 + d_0\, d_3)\,],$$
$$L^2{}_2 = 2[\, (b_0^2 - b_2^2) + (d_1^2 + d_3^2)\,] - 1, \qquad L^2{}_3 = 2[-(b_2\, b_3 + d_2\, d_3) - i\, (b_0\, b_1 + d_0\, d_1)\,],$$
$$L^3{}_0 = 2[-(b_1\, d_2 - b_2\, d_1) + i\, (b_0\, d_3 - b_3\, d_0)\,], \quad L^3{}_1 = 2[-\, (b_1\, b_3 + d_1\, d_3) - i\, (b_0\, b_2 + d_0\, d_2)\,],$$
$$L^3{}_2 = 2[-(b_2\, b_3 + d_2\, d_3) + i\, (b_0\, b_1 + d_0\, d_1)\,], \quad L^3{}_3 = 2[\, (b_0^2 - b_3^2) + (d_1^2 + d_2^2)\,] - 1,$$

which allow to obtain L if we have the expansion (5). However, here we have the inverse problem, that is, to obtain b_μ & d_μ, $\mu = 0, \dots, 3$ verifying (7) for a given Lorentz matrix. Our answer is the following:

$$b_0 = \frac{1}{4}\left(\alpha + \bar{\alpha} + \delta + \bar{\delta}\right),\ b_1 = \frac{1}{4}\left(\bar{\beta} - \beta + \bar{\gamma} - \gamma\right),\ b_2 = \frac{i}{4}\left(\beta + \bar{\beta} - \gamma - \bar{\gamma}\right),\ b_3 = \frac{1}{4}\left(\bar{\alpha} - \alpha + \delta - \bar{\delta}\right),$$
$$(8)$$
$$d_0 = \frac{i}{4}\left(\alpha - \bar{\alpha} + \delta - \bar{\delta}\right),\ d_1 = -\frac{i}{4}\left(\bar{\beta} + \beta + \bar{\gamma} + \gamma\right),\ d_2 = \frac{1}{4}\left(\bar{\beta} - \beta + \gamma - \bar{\gamma}\right),\ d_3 = \frac{i}{4}\left(\bar{\delta} + \delta - \alpha - \bar{\alpha}\right),$$

hence the expressions (1) are deduced if we apply (8) into (7). Besides, with (8) the matrix (5) acquires the structure:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \qquad A = \frac{1}{2}\begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \qquad E = \frac{1}{2}\begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \bar{\gamma} + \beta & \bar{\delta} - \alpha \end{pmatrix}. \tag{9}$$

Therefore, for a given Lorentz transformation first we employ (2) to determine $\alpha, \beta, \gamma, \delta$, then *S* is immediate via (9); this approach is an alternative to the process showed in [Caicedo-Ortiz, H. E. *et al.*, 2021] and to the explicit general formula obtained by Macfarlane [Macfarlane, A. J. 1966]:

$$S = \frac{1}{4\sqrt{G}}\, [G\, I + \frac{i}{2}\, \varepsilon_{\mu\nu\alpha\beta}\, L^{\mu\nu}\, L^{\alpha\beta}\, \gamma^5 + i\, \Gamma(L^2) - i\, (2 + tr\, L)\, \Gamma(L)], \tag{10}$$

such that:

$$G = 2\, (1 + tr\, L) + \frac{1}{2}[\, (tr\, L)^2 - tr\, L^2\,], \qquad tr\, L = \sum_{\mu=0}^{3} L^{\mu}{}_{\mu}\,, \qquad tr\, L^2 = \sum_{\nu,\alpha=0}^{3} L^{\nu}{}_{\alpha}\, L^{\alpha}{}_{\nu}\,,$$
$$(11)$$
$$\Gamma(L) = \sum_{\mu,\nu=0}^{3} L_{\mu\nu}\, \sigma^{\mu\nu}, \qquad \Gamma(L^2) = \sum_{\alpha,\mu,\nu=0}^{3} L_{\mu\alpha}\, L^{\alpha}{}_{\nu}\, \sigma^{\mu\nu}.$$

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Unimodular Complex Matrix

The expressions (2) allow obtain the complex matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with the constraint $\alpha\delta - \beta\gamma = 1$, but we consider that it is important to study the complex quantity D in such relations. In fact, from (2):

$$\alpha + \delta = \frac{tr\,L}{D}, \qquad D^2 = \det Q = det \begin{pmatrix} Q^1{}_1 & Q^1{}_2 \\ Q^2{}_1 & Q^2{}_2 \end{pmatrix}, \tag{12}$$

and the application of (1) in (2) gives the properties:

$$Q^1{}_1 = (\bar{\alpha} + \bar{\delta})\,\alpha\,, \quad Q^1{}_2 = (\bar{\alpha} + \bar{\delta})\,\beta\,, \quad Q^2{}_1 = (\bar{\alpha} + \bar{\delta})\,\gamma\,, \quad Q^2{}_2 = (\bar{\alpha} + \bar{\delta})\,\delta\,, \tag{13}$$

that is, $Q = (\bar{\alpha} + \bar{\delta})\,B$, thus:

$$D^2 = (\bar{\alpha} + \bar{\delta})^2 = \frac{(tr\,L)^2}{\bar{D}^2} \quad \therefore \quad D\bar{D} = tr\,L\,. \tag{14}$$

On the other hand, from (2):

$$D^2 = Q^1{}_1\,Q^2{}_2 - Q^1{}_2\,Q^2{}_1 = \frac{1}{4}\{(tr\,L)^2 + (L^1{}_2 - L^2{}_1)^2 + (L^1{}_3 - L^3{}_1)^2 + (L^2{}_3 - L^3{}_2)^2 - (L^0{}_1 + L^1{}_0)^2 -$$
$$-(L^0{}_2 + L^2{}_0)^2 - (L^0{}_3 + L^3{}_0)^2 + 2i\,[(L^0{}_1 + L^1{}_0)(L^2{}_3 - L^3{}_2) + (L^0{}_3 + L^3{}_0)(L^1{}_2 - L^2{}_1) - \tag{15}$$
$$-(L^0{}_2 + L^2{}_0)(L^1{}_3 - L^3{}_1)]\},$$

then with (14) and (15) we calculate the following positive real quantity:

$$G \equiv (D + \bar{D})^2 = D^2 + \bar{D}^2 + 2\,D\,\bar{D} = 2\,(1 + tr\,L) + \frac{1}{2}[\,(tr\,L)^2 - tr\,L^2\,] \quad \therefore \quad D + \bar{D} = \sqrt{G}, \tag{16}$$

in agreement with (11). Similarly:

$$D - \bar{D} = i\,\sqrt{H}\,, \quad H = 2\,(tr\,L - 1) + \frac{1}{2}[\,tr\,L^2 - (tr\,L)^2\,] \quad \therefore \quad D = \frac{1}{2}\,[\sqrt{G} + i\,\sqrt{H}\,], \tag{17}$$

hence for a given Lorentz mapping we determine G, H, D and finally the relations (2) imply the corresponding values for $\alpha, \beta, \gamma, \delta$, whose application in (8) allows deduce the expressions:

$$b_0 = \frac{\sqrt{G}}{4}, \quad d_0 = \frac{\sqrt{H}}{4}, \quad b_1 = \frac{i}{4\,tr\,L}\,[\,(L^0{}_1 + L^1{}_0)\,\sqrt{H} + (L^2{}_3 - L^3{}_2)\,\sqrt{G}\,],$$
$$b_2 = \frac{i}{4\,tr\,L}\,[\,(L^0{}_2 - L^2{}_0)\,\sqrt{H} + (L^3{}_1 - L^1{}_3)\,\sqrt{G}\,], \quad b_3 = \frac{i}{4\,tr\,L}\,[\,(L^0{}_3 + L^3{}_0)\,\sqrt{H} + (L^1{}_2 - L^2{}_1)\,\sqrt{G}\,],$$
$$(18)$$
$$d_1 = \frac{i}{4\,tr\,L}\,[\,(L^2{}_3 - L^3{}_2)\,\sqrt{H} - (L^0{}_1 + L^1{}_0)\,\sqrt{G}\,], \quad d_2 = \frac{i}{4\,tr\,L}\,[\,(L^3{}_1 - L^1{}_3)\,\sqrt{H} - (L^0{}_2 + L^2{}_0)\,\sqrt{G}\,],$$
$$d_3 = \frac{i}{4\,tr\,L}\,[\,(L^1{}_2 - L^2{}_1)\,\sqrt{H} - (L^0{}_3 + L^3{}_0)\,\sqrt{G}\,],$$

and (5) gives the matrix S in terms of the gamma matrices in the Dirac-Pauli representation. The relations (18) are compatible with the results (10) and (11) obtained by Macfarlane [Macfarlane, A. J. 1966].

CONCLUSIONS

For a given Lorentz transformation, our analysis gives its associated unimodular complex matrix and also the matrix that transforms the Dirac 4-spinor. From (5) and (10) we see that S is a linear combination of eight gamma matrices: I, γ^5 and $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$.

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