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Two Special Quaternions

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Abstract: For a given Lorentz matrix, we construct two special quaternions which generate the governing matrix to the Dirac spinor's transformation and its expansion in terms of the gamma matrices.
Keywords: Dirac spinor, Lorentz transformation, Unimodular matrix, Dirac matrices.

INTRODUCTION

The arbitrary complex quantities $\alpha, \beta, \gamma, \delta$ verifying the condition $\alpha\delta - \beta\gamma = 1$, generate a Lorentz matrix $L = (L^{\mu}{}_{\nu})$ via the expressions (Rumer, J. 1936; Aharoni, J. 1959; Synge, J. L. 1965; Penrose, R. and Rindler, W. 2002; López-Bonilla, J. *et al.*, 2002; Acevedo, M. *et al.*, 2005; Ahsan, Z. *et al.*, 2014; López-Bonilla, J. and Morales-García, M. 2020; Cruz-Santiago, R. *et al.*, 2021).

$$L^0{}_0 = \tfrac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}), \qquad L^1{}_0 = \tfrac{1}{2}(\bar{\alpha}\,\gamma + \bar{\beta}\delta) + cc, \qquad L^2{}_0 = -\tfrac{i}{2}(\alpha\bar{\gamma} - \bar{\beta}\delta) + cc\,,$$

$$L^0{}_1 = \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \qquad L^1{}_1 = \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \qquad L^2{}_1 = -\frac{i}{2}(\alpha\bar{\delta} + \beta\bar{\gamma}) + cc,$$

$$L^0{}_2 = -\tfrac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \qquad L^1{}_2 = -\tfrac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \qquad L^2{}_2 = \tfrac{1}{2}(\bar{\alpha}\delta - \bar{\beta}\gamma) + cc\,, \quad (1)$$

$$L^0{}_3 = \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}), \qquad L^1{}_3 = \frac{1}{2}(\bar{\alpha}\gamma - \bar{\beta}\delta) + cc, \qquad L^2{}_3 = -\frac{i}{2}(\alpha\bar{\gamma} + \bar{\beta}\delta) + cc\,,$$

$$L^3{}_0 = \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}), \qquad L^3{}_1 = \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \qquad L^3{}_2 = -\frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc\,,$$

$$L^3{}_3 = \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}), \qquad \alpha\delta - \beta\gamma = 1,$$

Where *cc* means the complex conjugate of all the previous terms.

The inverse problem is to obtain $\alpha, \beta, \gamma, \delta$ if we know L, and the answer is (Cruz-Santiago, R. *et al.*, 2021; Gürsey, F. 1955 Gürsey, F. 1957; Muller-Kirsten, H. J. and Wiedemann, A. 2010).

$$\alpha = \frac{1}{D}\,Q^1{}_1 = \frac{1}{2D}[L^0{}_0 + L^0{}_3 + L^1{}_1 + L^2{}_2 + L^3{}_0 + L^3{}_3 - i\,(L^1{}_2 - L^2{}_1)],$$

$$\beta = \frac{1}{D}\,Q^1{}_2 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 - L^1{}_3 + L^3{}_1 + i\,(L^0{}_2 + L^2{}_0 - L^2{}_3 + L^3{}_2)],$$

$$\gamma = \frac{1}{D}\,Q^2{}_1 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 + L^1{}_3 - L^3{}_1 - i\,(L^0{}_2 + L^2{}_0 + L^2{}_3 - L^3{}_2)], \quad (2)$$

$$\delta = \frac{1}{D}\,Q^2{}_2 = \frac{1}{2D}[L^0{}_0 - L^0{}_3 + L^1{}_1 + L^2{}_2 - L^3{}_0 + L^3{}_3 + i\,(L^1{}_2 - L^2{}_1)],$$

$$\text{where } D^2 = Q^1{}_1\,Q^2{}_2 - Q^1{}_2\,Q^2{}_1\,.$$

Dirac Spinor

The Dirac spinor obeys the transformation law (Leite-Lopes, J. 1977; Ohlsson, T. 2021).

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$$\tilde{\psi} = S\,\psi\,. \tag{3}$$

For a non-singular matrix S such that:

$$L^{\mu}{}_{\nu}\, S\, \gamma^{\nu} = \gamma^{\mu}\, S\,, \tag{4}$$

And we must determine a solution of (4) for a given Lorentz transformation.

Quaternions

If we know the Lorentz transformation, then first we employ (2) to determine $\alpha, \beta, \gamma, \delta$, and to construct the quaternions:

$$\boldsymbol{P} = p_0 + p_1\,\boldsymbol{I} + p_2\,\boldsymbol{J} + p_3\,\boldsymbol{K}, \qquad \boldsymbol{Q} = q_0 + q_1\,\boldsymbol{I} + q_2\,\boldsymbol{J} + q_3\,\boldsymbol{K}, \tag{5}$$

Such that:

$$p_0 = \tfrac{1}{4}(\alpha+\delta+\bar{\alpha}+\bar{\delta}),\; p_1 = \tfrac{i}{4}(\beta+\gamma-\bar{\beta}-\bar{\gamma}),\; p_2 = \tfrac{1}{4}(\gamma-\beta+\bar{\gamma}-\bar{\beta}),\; p_3 = \tfrac{i}{4}(\alpha-\delta-\bar{\alpha}+\bar{\delta}), \tag{6}$$

$$q_0 = \frac{1}{4}(-\alpha-\delta+\bar{\alpha}+\bar{\delta}), q_1 = -\frac{i}{4}(\beta+\gamma+\bar{\beta}+\bar{\gamma}), q_2 = \frac{1}{4}(-\gamma+\beta+\bar{\gamma}-\bar{\beta}), q_3 = \frac{i}{4}(-\alpha+\delta-\bar{\alpha}+\bar{\delta}),$$

The quaternions (5) generate the following complex matrices:

$$\boldsymbol{P} \;\rightarrow\; A = \tfrac{1}{2}\begin{pmatrix} \delta+\bar{\alpha} & -\gamma+\bar{\beta} \\ -\beta+\bar{\gamma} & \alpha+\bar{\delta}\end{pmatrix}, \qquad \boldsymbol{Q} \;\rightarrow\; E = \tfrac{1}{2}\begin{pmatrix} -\delta+\bar{\alpha} & \gamma+\bar{\beta} \\ \beta+\bar{\gamma} & -\alpha+\bar{\delta}\end{pmatrix}, \tag{7}$$

And the surprise is that (7) gives the solution to (4), in fact:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \tag{8}$$

For a given Lorentz mapping. Besides, if S is written in terms of Dirac matrices, in the standard representation (Leite-Lopes, J. 1977; Caicedo-Ortiz, H. E. *et al.*, 2021),

Then (6) are the coefficients in the corresponding expansion:

$$S = p_0\, I + i\, p_1\, \sigma^{23} - i\, p_2\, \sigma^{31} + i\, p_3\, \sigma^{12} + q_0\, \gamma^{5} + q_1\, \sigma^{01} - q_2\, \sigma^{02} + q_3\, \sigma^{03}. \tag{9}$$

CONCLUSIONS

The Dirac equation has relativistic structure if the Dirac spinor verifies the law transformation (3), then for a given Lorentz transformation it is necessary to obtain S in according with (4), and here we exhibit the special quaternions (5) which allow construct S in the general case.

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