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Three applications of hypergeometric functions

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Abstract: We give a simple proof of a relation obtained by Qi-Guo for the sum of the Lah numbers in terms of the confluent hypergeometric function. We deduce the hypergeometric version of a combinatorial identity obtained by Engbers-Stocker. Besides, we show an alternative deduction for the Rathie-Paris formula satisfied by the Gauss hypergeometric function.

Keywords: Hypergeometric function, Petkovsek-Wilf-Zeilberger's method, Lah numbers, Binomial coefficient, Incomplete Beta function.

INTRODUCTION

In the literature we find many series studied with complicated methods, which could be analyzed more easily with hypergeometric techniques; then here we apply these techniques, involving the functions ${}_2F_1$ and ${}_3F_2$, to the identities obtained by Qi-Guo, Engbers-Stocker, Alzer-Prodinger, and Rathie-Paris.

Qi-Guo, Engbers-Stocker and Rathie-Paris identities

Qi-Guo (Qi, F. and Bai-Ni, G. 2018; luschny.de/) showed the following relation:

$$A \equiv \sum_{k=1}^{n} L_n^{[k]}\, z^{k-1} = n!\; e^{-z}\; {}_1F_1(n+1; 2; z), \qquad (1)$$

Involving the Lah numbers (Lah, I. 1954; Aigner, M. 2007; Daboul, S., J. *et al.*, 2013; Boyadzhiev, K. N. 2016; Spivey, M. Z. 2019; López-Bonilla, J. and Montiel-Pérez, J.Y. 2020; López-Bonilla, J. *et al.*, 2020).

$$L_n^{[k]} = \frac{n!}{k!}\binom{n-1}{k-1}, \qquad (2)$$

And the confluent hypergeometric function (mathworld.wolfram.com); here we shall prove (1) via the algorithm explained in (Petkovsek, M. *et al.*, 1996; Koepf, W. 1998; Koepf, W. 2007; Hannah, J. P. 2013; Guerrero-Moreno, I. and López-Bonilla, J. 2016; López-Bonilla, J. *et al.*, 2018; Barrera-Figueroa, V. *et al.*, 2018; Léon-Vega, C. *et al.*, 2018; López-Bonilla, J. and Miranda-Sánchez, I. 2020). In fact, from (2):

$$A = n!\sum_{r=0}^{\infty} t_r\,, \qquad t_r = \frac{(n-1)!}{r!\,(r+1)!\,(n-r-1)!}\, z^r, \qquad (3)$$

Therefore $\frac{t_{r+1}}{t_r} = \frac{(r+1-n)}{(r+2)(r+1)}\,(-z)$, then (3) implies the hypergeometric relation:

$$A = n!\;{}_1F_1(1-n; 2; -z), \qquad (4)$$

But we have the Kummer's identity (mathworld.wolfram.com):

$${}_1F_1(a; b; -z) = e^{-z}\;{}_1F_1(b-a; b; z), \qquad (5)$$

Whose application in (4) gives (1), q.e.d.

Engbers-Stocker [20, 21] showed the following combinatorial identity:

$$B \equiv \sum_{k=0}^{m}\binom{m}{k}^2\binom{n+1+k}{2m+1} = \sum_{r=m}^{n}\binom{r}{m}^2, \qquad n \geq m \geq 0, \qquad (6)$$

And here we obtain its hypergeometric version for the case $n \geq 2m$ via the method explained in (Petkovsek, M. *et al.*, 1996; Koepf, W. 1998; Koepf, W. 2007; Hannah, J. P. 2013; Guerrero-Moreno, I. and López-Bonilla, J. 2016; López-Bonilla, J. *et al.*, 2018; Barrera-Figueroa, V. *et al.*,

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2018; Léon-Vega, C. *et al.*, 2018; López-Bonilla, J. and Miranda-Sánchez, I. 2020). In fact:

$$B = \binom{n+1}{2m+1} \sum_{k=0}^{\infty} b_k \,, \qquad b_k = \frac{\binom{m}{k}^2 \binom{n+1+k}{2m+1}}{\binom{n+1}{2m+1}},$$

Thus:

$$\frac{b_{k+1}}{b_k} = \frac{(k - m)^2 \, (k + n + 2)}{(k + 1)^2 \, (k + n + 1 - 2m)},$$

Therefore:

$$B = \binom{n+1}{2m+1} \; {}_3F_2(-m, -m, n+2; 1, n+1-2m; 1), \qquad (7)$$

Then (6) and (7) imply the hypergeometric relation:

$${}_3F_2(-m, -m, n+2; 1, n+1-2m; 1) = \frac{1}{\binom{n+1}{2m+1}} \sum_{r=m}^{n} \binom{r}{m}^2, \qquad n \geq 2m \geq 0; \qquad (8)$$

In particular, for $n = 2m$:

$${}_3F_2(-m, -m, 2m+2; 1, 1; 1) = \sum_{r=m}^{2m} \binom{r}{m}^2, \quad m \geq 0. \qquad (9)$$

Alzer-Prodinger (Engbers, J. and Ch. Stocker. 2016) generalized the expression (6):

$$\sum_{k=0}^{m} \binom{m}{k}^2 \binom{n+1+k}{2m+N} = \sum_{r=m}^{n} \binom{r}{m}^2 \binom{n-r}{N-1}, \qquad n \geq m \geq 0, \quad N \geq 1, \qquad (10)$$

Which gives the following generalization of (8):

$${}_3F_2(-m, -m, n+2; 1, n+2-2m-N; 1) = \frac{1}{\binom{n+1}{2m+N}} \sum_{r=m}^{n} \binom{r}{m}^2 \binom{n-r}{N-1}, \; n \geq 2m+N-1, \qquad (11)$$

Thus (10) and (11) imply (6) and (8) if $N = 1$.

From (11) for $n = 2m + N - 1$:

$${}_3F_2(-m, -m, 2m+N+1; 1, 1; 1) = \sum_{r=m}^{2m+N-1} \binom{r}{m}^2 \binom{2m+N-r-1}{N-1}, \quad m \geq 0, \quad N \geq 1, \qquad (12)$$

Which gives (8) for the case $N = 1$.

Rathie-Paris (Rathie, A. K. and Paris, R. B. 2007) obtained the following relation for $0 < x < 1$:

$$G_n(x) \equiv x \; {}_2F_1\left(-n, 1; -2n; \tfrac{1}{1-x}\right) + (1-x) \; {}_2F_1\left(-n, 1; \, -2n; \tfrac{1}{x}\right) = \frac{(n!)^2}{(2n)! \, [x\,(1-x)]^n}, \qquad (13)$$

Involving the Gauss hypergeometric function. Here we show other approach towards this identity.

In (unctions.wolfram.com) is the expression:

$${}_2F_1(-n, 1; c; z) = (1-c) \, z^{1-c} \, (z-1)^{c+n-1} \, B_{1-\frac{1}{z}}(1-c-n, n+1), \qquad (14)$$

With the participation of the incomplete Beta function:

$$B_w(a, b) = \int_0^w t^{a-1} \, (1-t)^{b-1} \, dt. \qquad (15)$$

Therefore:

$$G_n(x) = \frac{2n+1}{[x\,(1-x)]^n} \, H_n(x), \qquad (16)$$

Where:

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$$H_n(x) \equiv B_x(n+1,n+1) + B_{1-x}(n+1,n+1) = \int_0^x [t(1-t)]^n \, dt + \int_0^{1-x} [t(1-t)]^n \, dt, \qquad (17)$$

Then it is immediate that $\frac{d}{dx} H_n(x) = 0$, thus (17) is independent of x and we can calculate $H_n(x)$ with any value of x, hence we take $x = \frac{1}{2}$:

$$H_n\left(\frac{1}{2}\right) = 2\, B_{\frac{1}{2}}(n+1,n+1) = 2 \int_0^{\frac{1}{2}} [t(1-t)]^n \, dt. \qquad (18)$$

From (14) and [24]:

$$B_{\frac{1}{2}}(n+1,n+1) = \frac{1}{(2n+1)\, 2^{2n+1}} \; {}_2F_1(-n, 1; -2n; 2), \qquad {}_2F_1(-n,1;-2n;2) = \frac{2^{2n}\,(n!)^2}{(2n)!}, \qquad (19)$$

Whose application in (18) gives the expression?

$$H_n(x) = \frac{(n!)^2}{(2n+1)!}, \qquad (20)$$

Then (16) and (20) imply (13), q.e.d.

CONCLUSIONS

Our procedure shows that the method of Petkovsek-Wilf-Zeilberger (Petkovsek, M. *et al.*, 1996; Koepf, W. 1998; Koepf, W. 2007; Hannah, J. P. 2013; Guerrero-Moreno, I. and López-Bonilla, J. 2016; López-Bonilla, J. *et al.*, 2018; Barrera-Figueroa, V. *et al.*, 2018; Léon-Vega, C. *et al.*, 2018; López-Bonilla, J. and Miranda-Sánchez, I. 2020; Gasper, G. and M. Rahman. 2005) allows give a simple demonstration for the Qi-Guo's identity (1) and also obtain the property (8) for the hypergeometric function ${}_3F_2$ from the Engbers-Stocker's formula (6), with the corresponding generalizations (10) and (11). Finally, properties of ${}_2F_1$ and incomplete Beta function gives us an alternative proof for the Rathie-Paris's relation (13).

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